Review Day

- Pls ask questions!!
- Also, we reached 85% on the SETS!!

Wahoo Yay

What is a basis? Let $(v_1 \dots v_n)$ be in V

- $v_1 \dots v_n$ are LI
- $\operatorname{span}(v_1 \dots v_n) = V$

ex) $\mathbb{R}^n$ - Q: What is max # of LI vectors?

A: n

- to check if $\underbrace{v_1 \dots v_k}_{\text{in } \mathbb{R}^n}$ are LI

$$\longrightarrow \begin{pmatrix} v_1 & v_2 & \cdots & v_k \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

Q: What is the min # of spanning vectors

A: n

- to check if $\underbrace{v_1 \ldots v_k}_{\text{in } \mathbb{R}^n}$ span $\mathbb{R}^n$

$$\longrightarrow \begin{pmatrix} v_1 & \cdots & v_k \\ \downarrow & & \downarrow \end{pmatrix}$$

ex) Is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$ a basis for $\mathbb{R}^3$

No – there are 4 vectors

Can check if they span

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

neither spanning/LI

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Spanning not LI

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

LI not spanning

$$\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}_{3\times 3}$$

LI but not spanning?
Spanning but not LI?

impossible (half is good enough)

Coordinate vectors

Let $\mathcal{B} = (v_1 \ldots v_n)$ be a basis for V.

- then any w in V can be written uniquely as

$$w = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

Def: The coordinate vector with respect to basis $\mathcal{B}$ is

$$[w]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \text{ in } \mathbb{R}^n$$

ex) $V = \mathbb{R}_2[x]$ and $\mathcal{B} = (1, x, x^2)$

Q: Find $[2-4x+12x^2]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -4 \\ 12 \end{pmatrix}$

$[2-10x^2]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \\ -10 \end{pmatrix}$

$[x+x^2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$[1+x]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Q: Suppose f is in $\mathbb{R}_2[x]$ with

$[f]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 2 \\ -10 \end{pmatrix}$

What is f? $\underline{4 + 2x - 10x^2 = f}$

ex 2) $M_{2\times 2}(\mathbb{R})$ $\quad B = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

m is a matrix with

$[m]_B = \begin{pmatrix} 10 \\ -2 \\ 0 \\ -20 \end{pmatrix}$ in $\mathbb{R}^4$

$$m = 10\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - 20\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{m = \begin{pmatrix} 10 & -2 \\ 0 & -20 \end{pmatrix}}$$

Change of basis

V vector space $\quad \mathcal{B}_1 = (v_1 \dots v_n)$

$\mathcal{B}_2 = (w_1 \dots w_n)$

then for any z in V

- $[z]_{\mathcal{B}_1}$

- $[z]_{\mathcal{B}_2}$

Q) How are $[z]_{\mathcal{B}_1}$ and $[z]_{\mathcal{B}_2}$ related?

A: through change of basis matrix!

$$P_{\mathcal{B}_1 \to \mathcal{B}_2} = \begin{pmatrix} [v_1]_{\mathcal{B}_2} & [v_2]_{\mathcal{B}_2} & \cdots & [v_n]_{\mathcal{B}_2} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

"change of basis matrix (from $\mathcal{B}_1$ to $\mathcal{B}_2$)"

Name is justified because

$$[x]_{\mathcal{B}_2} = P_{\mathcal{B}_1 \to \mathcal{B}_2} [x]_{\mathcal{B}_1}$$

ex) $V = \mathbb{R}_2[x]$ $\quad B_1 = (1, x, x^2)$

$B_2 = (1, 1+x, 1+x^2)$

Q1) Find $[4 - 10x^2]_{B_1} = \begin{pmatrix} 4 \\ 0 \\ -10 \end{pmatrix}$

Q2) Find $P_{B_1 \to B_2}$

Q3) Find $[4 - 10x^2]_{B_2}$

Q2) $\underline{P_{B_1 \to B_2}}$

- $1 = a_1(1) + a_2(1+x) + a_3(1+x^2)$

  $= 1(1) + 0(1+x) + 0(1+x^2)$

$\Rightarrow [1]_{B_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

- $x = a_1(1) + a_2(1+x) + a_3(1+x^2)$

  $x = a_1 + a_2 + a_2 x + a_3 + a_3 x^2$

  $x = (a_1 + a_2 + a_3) + (a_2)x + (a_3)x^2$

$\Rightarrow \begin{array}{l} a_1 + a_2 + a_3 = 0 \\ a_2 = 1 \\ a_3 = 0 \end{array} \Rightarrow \begin{array}{l} a_1 = -1 \\ a_2 = 1 \\ a_3 = 0 \end{array}$

$$\Rightarrow [x]_{B_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

- $x^2 = a_1(1) + a_2(1+x) + a_3(1+x^2)$

$$x^2 = a_1 + a_2 + a_2 x + a_3 + a_3 x^2$$

$$x^2 = (a_1 + a_2 + a_3) + a_2 x + a_3 x^2$$

$$\Rightarrow a_1 = -1$$
$$a_2 = 0$$
$$a_3 = 1$$

$$\Rightarrow [x^2]_{B_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$P_{B_1 \to B_2} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Q3: Find $[4-10x^2]_{B_2}$

Have $[4-10x^2]_{B_2} = \underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{P_{B_1 \to B_2}} \underbrace{\begin{pmatrix} 4 \\ 0 \\ -10 \end{pmatrix}}_{[4-10x^2]_{B_1}}$

$$= 4\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - 10\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Claim: $[4-10x^2]_{B_2} = \begin{pmatrix} 14 \\ 0 \\ -10 \end{pmatrix}$

$$14(1) + 0(1+x) - 10(1+x^2) = 14 - 10 - 10x^2$$
$$= 4 - 10x^2 \quad \checkmark$$

$$\left(P_{B_1 \to B_2}\right)^{-1} = \underline{P_{B_2 \to B_1}}$$

$T : V \to W$ linear transformation

1) $T(x+y) = T(x) + T(y)$ for any x, y in V

2) $T(cx) = cT(x)$ for any x in V, c in $\mathbb{R}$.

$B_V = (v_1 \ldots v_n)$ basis for V

$T: V \to W$

$B_W = (w_1 \ldots w_m)$ basis for W

$$A_{T, B_V, B_W} = \begin{pmatrix} [T(v_1)]_{B_W} & [T(v_2)]_{B_W} & \cdots & [T(v_n)]_{B_W} \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix}_{m \times n}$$

For any z in V

$$\underbrace{[T(z)]_{B_W}}_{\text{in } \mathbb{R}^m} = \underbrace{A_{T, B_V, B_W}}_{m \times n} \underbrace{[z]_{B_V}}_{\text{in } \mathbb{R}^n}$$

$T: V \to W$	$A_{T,\beta_V,\beta_W} = A_{m \times n}$
$\text{Ker}(T)$	$\text{null}(A)$
Range $R(T)$ = image	$\text{col}(A) = \text{span}(\text{columns of } A)$
$\text{Rank}(T) = \dim(R(T))$	$\text{rank}(A) = \dim(\text{col}(A))$
$\text{nullity}(T) = \dim(\text{Ker}(T))$	$\text{nullity}(A) = \dim(\text{null}(A))$
is T injective?	Are columns LI?
is T surjective?	Do columns span $\mathbb{R}^m$?
is T an isomorphism?	Is A invertible?
$\dim V = \text{rank}(T) + \text{nullity}(T)$ $\parallel$ n	$\#\text{columns} = \text{rank}(A) + \text{nullity}(A)$
$\det(T)$	$\det(A_T)$

Suppose $T: \underset{V}{\mathbb{R}_2[x]} \longrightarrow \underset{W}{M_{2\times 2}(\mathbb{R})}$

$\mathcal{B}_V = (1, x, x^2)$

$\mathcal{B}_W = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

with $A_{T, \mathcal{B}_V, \mathcal{B}_W} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}_{4\times 3}$

Q: What is $T(1+x+x^2)$?

A: First find $[T(1+x+x^2)]_{\mathcal{B}_W}$

$$= A_{T,\mathcal{B}_V,\mathcal{B}_W}[1+x+x^2]_{\mathcal{B}_V}$$

$$[T(1+x+x^2)]_{\mathcal{B}_W} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= 1\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$[T(1+x+x^2)]_{\mathcal{B}_W} = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 3 \end{pmatrix}$$

What is $T(1+x+x^2)$?

$$T(1+x+x^2) = \begin{pmatrix} 2 & 3 \\ 6 & 3 \end{pmatrix}$$

$$T : \overset{V}{\overset{\shortparallel}{M_{2\times 2}(\mathbb{R})}} \longrightarrow \overset{W}{\overset{\shortparallel}{\mathbb{R}_4[x]}}$$

$\mathcal{B}_V, \mathcal{B}_W$ standard.

and $$A_{T,\mathcal{B}_V,\mathcal{B}_W} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}_{5\times 4}$$

What is $T\left(\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\right)$

• Find $\left[T\begin{pmatrix}2&3\\1&4\end{pmatrix}\right]_{B_W} = A_{T,B_V,B_W}\left[\begin{pmatrix}2&3\\1&4\end{pmatrix}\right]_{B_V}$

$$= \begin{pmatrix}1&1&3&0\\0&2&1&0\\1&1&0&1\\0&0&1&0\\1&0&1&0\end{pmatrix}\begin{pmatrix}2\\3\\1\\4\end{pmatrix}$$

$$= 2\begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix} + 3\begin{pmatrix}1\\2\\1\\0\\0\end{pmatrix} + 1\begin{pmatrix}3\\1\\0\\1\\1\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$$

$$\left[T\begin{pmatrix}2&3\\1&4\end{pmatrix}\right]_{B_W} = \begin{pmatrix}8\\7\\6\\1\\3\end{pmatrix}$$

$$\boxed{\text{So } T\begin{pmatrix}2&3\\1&4\end{pmatrix} = 8+7x+6x^2+x^3+3x^4}$$

Eigenvalues / vectors

$$T(v) = \lambda v$$

$$A\vec{v} = \lambda \vec{v}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ has eigenvalues

$\lambda_1 = 1 \rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = 1 \rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = 2 \rightarrow v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\det(A - \lambda I) = 0$$

Find null A

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 4}$$

x_2 = free = r

x_4 = free = s

$x_1 + r + 2(0) + s = 0$

$x_1 = -r - s$

$$\text{null}(A) = \begin{pmatrix} -r-s \\ r \\ 0 \\ s \end{pmatrix}$$

$$\text{null}(A) = \begin{pmatrix} -r \\ r \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -s \\ 0 \\ 0 \\ s \end{pmatrix}$$

$$\text{null}(A) = r\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\text{null}(A) = \text{span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)}$$